Exercise Sheet 7

Exercise 1. Let $b, c \in \mathbb{C}$ with $\Delta = b^2 - 4c \neq 0$. Consider the projective curve C defined by

$$Y^2 = X^2 + bXZ + cZ^2.$$

(1) Show that C is smooth.
(2) On the affine chart $Z = 1$ (so $x = X/Z,\ y = Y/Z$) define $t := y - x$. Solve for x and y in terms of t, and deduce a rational parametrization $t \mapsto (x(t), y(t))$ of C. Show that this extends to a biholomorphism $\Phi : \mathbb{P}^1 \to C$. **Hint:** Homogenize $t = u/v$ and verify that

$$\Phi([u : v]) = \left[cv^2 - u^2 : u^2 - buv + cv^2 : 2uv - bv^2\right]$$

has inverse $\Psi([X : Y : Z]) = [Y - X : Z]$.
(3) Use the rational parametrization from (2) to deduce Euler's formula

$$\int \frac{dx}{\sqrt{x^2 + bx + c}} = -\log|2t - b| + C,$$

with $t = \sqrt{x^2 + bx + c} - x$.

Exercise 2. Let $F(X, Y, Z) = ZY^2 - X^3$ and consider the associated projective curve $C \subset \mathbb{P}^2$. Let $\nu : \tilde{C} \to C$ be the normalization of C.

(1) What are the singular points of C?
(2) Show that the normalization $\tilde{C}$ is biholomorphic to $\mathbb{P}^1$.

Exercise 3. (for credit, due on 16 November)
Let $n \geq 4$ be even and $a_1, \ldots, a_n \in \mathbb{C}$ be nonzero and distinct. Consider the projective curve $C \subset \mathbb{P}^2$ defined by

$$F(X, Y, Z) = Y^2 Z^{n-2} - \prod_{i=1}^{n} (X - a_i Z) = 0.$$

(1) *(1 point)* Show that $P_\infty = [0 : 1 : 0]$ is the only singular point of C.

Let $\nu : \tilde{C} \to C$ be the normalization of C. Consider the holomorphic map $\pi : \tilde{C} \to \mathbb{P}^1$ defined by

$$Q \mapsto \begin{cases} [X : Z] & \text{if } Q = [X : Y : Z] \in C \setminus \{P_\infty\} \cong \tilde{C} \setminus \nu^{-1}(P_\infty); \\ [1 : 0] & \text{if } Q \in \nu^{-1}(P_\infty). \end{cases}$$

(2) *(1 point)* Compute the ramification points of π on $C \setminus \{P_\infty\} \cong \tilde{C} \setminus \nu^{-1}(P_\infty)$.

On $\tilde{C}$ we define the meromorphic functions

$$u = \left(\frac{Z}{X}\right) \circ \nu, \quad v = \left(\frac{YZ^{n/2-1}}{X^{n/2}}\right) \circ \nu.$$

(3) *(1 point)* Show that for every $Q \in \nu^{-1}(P_\infty)$ we have $u(Q) = 0$. **Hint:** Work in the chart $Y \neq 0$ and set $\xi = X/Y$ and $\zeta = Z/Y$. Deduce from the curve equation $1/\zeta^2 = \prod(1/u - a_i)$ that $u \to 0$ as we approach $(\xi, \zeta) = (0, 0)$.
(4) *(1 points)* Show that we can write the curve equation as

$$v^2 = \prod_{i=1}^{n} (1 - a_i u) =: G(u),$$

and deduce that at any $Q \in \tilde{C}$ with $u(Q) = 0$ we must have $v(Q) \in \{+1, -1\}$.

We now argue why the fiber $\pi^{-1}([1:0])$ actually consists of these two points. Let D be the affine curve D defined by

$$\Phi(u,v) = v^2 - G(u) = 0.$$

At $(u,v) = (0,\pm 1)$ we have $\Phi(0,\pm 1) = 0$ and $\partial_v \Phi(0,\pm 1) = 2(\pm 1) \neq 0$. By the implicit function theorem there exist a neighborhood U around 0 and holomorphic branches $v = \phi_\pm(u)$ on U with $\phi_\pm(0) = \pm 1$ and $\Phi(u, \phi_\pm(u)) = 0$. Thus, near $u = 0$, D is the disjoint union of the two graphs $v = \phi_+(u)$ and $v = \phi_-(u)$. Intersecting with the line $u = 0$ gives exactly the two points $(0,1)$ and $(0,-1)$ on D, which are smooth. Normalization is a biholomorphism over the smooth locus, hence there are $Q_\pm \in \widetilde{C}$ mapping to $(0,\pm 1)$. Therefore the fiber $\pi^{-1}([1:0])$ consists of two points $Q_\pm$, both of which are unramified (since the degree is 2).

(5) *(1 point)* Apply Riemann-Hurwitz to π to determine $g(\tilde{C})$.

Exercise 4. Determine the fundamental group of a closed surface of genus g with $n \geq 1$ distinct punctures.